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A network efficiency measure with application to critical infrastructure networks

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Abstract In this paper, we demonstrate how a new network performance/efficiency measure, which captures demands, flows, costs, and behavior on networks, can be used to assess the importance of network components and their rankings. We provide new results regarding the measure, which we refer to as the Nagurney–Qiang measure, or, simply, the N–Q measure, and a previously proposed one, which did not explicitly consider demands and flows. We apply both measures to such critical infrastructure networks as transportation networks and the Internet and further explore the new measure through an application to an electric power generation and distribution network in the form of a supply chain. The Nagurney and Qiang network performance/efficiency measure that captures flows and behavior can identify which network components, that is, nodes and links, have the greatest impact in terms of their removal and, hence, are important from both vulnerability as well as security standpoints.

Keywords Network efficiency measure · Network component importance ranking · Braess Paradox · Transportation networks · Internet · Electric power supply chain networks · Infrastructure networks · Network vulnerability · Critical infrastructure protection

1 Introduction

Recent events such as Hurricane Katrina, which hit the United States in August 2005, as well as the biggest blackout in North American history, which occurred in August 2004, and which was followed by power outages in Italy, as well as 9/11, have all demonstrated the importance as well as the vulnerability of critical infrastructure networks, in the form of transportation networks, electric power generation and distribution networks and, of course, the Internet, and even financial networks.

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In order to be able to assess the performance/efficiency of a network it is imperative that appropriate measures be devised since only when the performance of a network can be quantifiably measured can the network be appropriately managed. In addition, appropriate network measures can assist in the identification of the importance of network components, that is, nodes and links, and their rankings. Such rankings can be very helpful in the case of the determination of network vulnerabilities as well as when to reinforce/enhance security.

In this paper, a recently proposed new network efficiency measure by Nagurney and Qiang (cf. [15]) is applied to several critical infrastructure networks in order to ascertain the efficiency of the networks as well as the importance and the ranking of the network components. The importance and the ranking of the network components are determined for a variety of networks in order to illustrate and emphasize that demands and flows, and incurred costs, are important factors in evaluating the importance of network components.

In addition, the results regarding the importance of network components as determined by the Nagurney–Qiang measure are compared to those obtained via the measure proposed by Latora and Marchiori [8]. For simplicity, in this paper, we refer to the Latora–Marchiori measure as the L–M measure and to the Nagurney–Qiang measure as the N–Q measure. The L–M measure, unlike the N–Q measure, does not take demands (and associated flows and behavior) into consideration. The L–M measure has been widely applied (cf. [9,10]), especially by physicists, and in the context of what are commonly referred to as *complex* networks (see Newman [19] and the references therein).

Furthermore, we note that, recently, Jenelius et al. [7] proposed several link importance indicators and applied them to the road transportation network in northern Sweden. In particular, they proposed distinct link importance indicators depending upon whether the removal of the link would cause the network to become disconnected or not. It is worth pointing out that their measures also take costs and flows into consideration when evaluating a link's importance. However, as will be shown in the concrete examples in Sects. 4–6, the N–Q measure is a unified measure that can be applied to any network component, be it a node, or a link, or a set of nodes and links, and it is independent of whether the removal of links or nodes would cause the network to become disconnected or not. Murray-Tuite and Mahmassani [12] also focused on identifying indices for the determination of vulnerable links in transportation networks but our measure is unified and can be applied to identify the importance of either nodes or links or both.

This paper is organized as follows. In Sect. 2, we briefly recall the well-known fixed demand network equilibrium problem, due to Dafermos [5] and Smith [22], state the governing equilibrium conditions, and present the variational inequality formulation. Section 3 then reviews both the L–M and the N–Q network performance/efficiency measures and presents new theoretical results regarding their relationships. Section 4 revisits the Braess Paradox [3], which is as relevant to the Internet (see also, e.g., [14,18,21]) as it is to transportation networks, and applies both the N–Q and the L–M network measures to determine the efficiency of the network as well as the importance identification of the nodes and links (along with their rankings). Section 5 then considers a coupled Braess network and further illustrates why the N–Q measure provides more realistic results than the L–M measure. Section 6 concludes with an application to an electric power supply chain network, which is reformulated as a transportation network equilibrium problem over an appropriately constructed supernetwork, as discussed in [17]. Section 7 summarizes the results of this paper and presents our conclusions.

2 Network equilibrium model with fixed demands

Consider a network $G = [N, L]$ where N denotes the set of nodes with n elements and L the set of links with n_L elements. We assume that the links are directed. Let W denote the set of origin/destination (O/D) pairs of nodes in the network with a typical O/D pair denoted by w. Links are denoted by a, b, etc., and paths by p, q, etc. A path is assumed to be acyclic, and consists of a sequence of links that join an origin/destination pair of nodes. We denote the set of paths joining O/D pair w by P_w and the set of all paths joining all O/D pairs by P. There are n_P paths in the network.

We assume that the demand d_w is known and fixed for all O/D pairs $w \in W$. We denote the nonnegative flow on path p by x_p and the flow on link a by f_a and we group the path flows into the vector $x \in R_+^{n_P}$ and the link flows into the vector $f \in R_+^{n_L}$.

The following conservation of flow equations must hold:

$$\sum_{p \in P_w} x_p = d_w, \quad \forall w \in W, \tag{1}$$

which means that the sum of path flows on paths connecting each O/D pair must be equal to the demand for that O/D pair.

The link flows are related to the path flows, in turn, through the following conservation of flow equations:

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in L, \tag{2}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and $\delta_{ap} = 0$, otherwise. Hence, the flow on a link is equal to the sum of the flows on paths that contain that link.

The user cost on a path p is denoted by C_p and the user cost on a link a by c_a. The user costs on paths are related to user costs on links through the following equations:

$$C_p = \sum_{a \in L} c_a \delta_{ap}, \quad \forall p \in P, \tag{3}$$

that is, the user cost on a path is equal to the sum of user costs on links that make up the path.

For the sake of generality, we allow the user link cost function on each link to depend upon the entire vector of link flows, so that

$$c_a = c_a(f), \quad \forall a \in L. \tag{4}$$

We assume that the user link cost functions are continuous. In view of (1), (2), (3), and (4), we may write

$$C_p = C_p(x), \quad \forall p \in P. \tag{5}$$

Definition 1 (*Network equilibrium*) A path flow pattern $x^* \in \mathcal{K}^1$, where $\mathcal{K}^1 \equiv \{x | x \in R_+^{n_P}$ and (1) holds$\}$, is said to be a network equilibrium, if the following conditions hold for each O/D pair $w \in W$ and each path $p \in P_w$:

$$C_p(x^*) \begin{cases} = \lambda_w, & \text{if } x_p^* > 0, \\ \geq \lambda_w, & \text{if } x_p^* = 0. \end{cases} \tag{6}$$

The interpretation of conditions (6) is that all used paths connecting an O/D pair w have equal and minimal costs (with the minimal path costs equal to the equilibrium travel disutility, denoted by λ_w). These conditions are also referred to as the user-optimized conditions (cf. [6]) and correspond to Wardrop's [23] first principle of travel behavior. As established in [22] and [5], the equilibrium pattern according to Definition 1 is also the solution to the following variational inequality problem.

Theorem 1 *A path flow pattern $x^* \in \mathcal{K}^1$ is a network equilibrium according to Definition 1 if and only if it satisfies the variational inequality problem: determine $x^* \in \mathcal{K}^1$ such that*

$$\sum_{w\in W}\sum_{p\in P_w} C_p(x^*) \times \left[x_p - x_p^*\right] \geq 0, \quad \forall x \in \mathcal{K}^1. \tag{7}$$

Existence of a solution to variational inequality (7) follows from the standard theory of variational inequalities, under the assumption that the user link cost functions are continuous, since the feasible set $\mathcal{K}^1$ is compact (cf. [13]). Also, according to the theory of variational inequalities, uniqueness of an equilibrium link flow pattern, in turn, is then guaranteed, provided that the user link cost functions are strictly monotone. Algorithms for the solutions of variational inequality (7) can be found in [2,13,16], and the references therein; see also [20]. In the classical network equilibrium problem, in which the cost on each link (cf. (4)) depends solely on the flow on that link, the network equilibrium conditions (6) can be reformulated as the solution to an appropriately constructed optimization problem, as established in [1]; see also [6].

3 The N–Q measure vs. the L–M measure with new results

In this section, we recall two distinct network performance/efficiency measures and clarify their relationships.

Recall that the L–M measure (cf. [8]), which was proposed to measure the efficiency of networks in which the links may have associated weights or costs, is defined as follows:

Definition 2 (*The L–M measure*) The L–M network performance/efficiency measure, $E(G)$, according to [8] for a given network topology G, is defined as:

$$E(G) = \frac{1}{n(n-1)} \sum_{i\neq j\in G} \frac{1}{d_{ij}}, \tag{8}$$

where n is the number of nodes in the network and d_{ij} is the shortest path length between node i and node j.

The N–Q measure (cf. [15]), on the other hand, is defined in the context of network equilibrium, and it captures prices and costs and the underlying behavior of "users" of the network. The formal definition is as follows:

Definition 3 (*The N–Q measure*) The N–Q network performance/efficiency measure, $\mathcal{E}(G,d)$, according to [15], for a given network topology G and fixed demand vector d, is defined as:

$$\mathcal{E}(G,d) = \frac{\sum_{w\in W} \frac{d_w}{\lambda_w}}{n_W}, \tag{9}$$

where recall that n_W is the number of O/D pairs in the network and λ_w is the equilibrium disutility for O/D pair w (cf. (6)).

Theorem 2 *If positive demands exist for all pairs of nodes in the network G, and each of these demands is equal to* 1 *and if d_{ij} is set equal to λ_w, where $w = (i, j)$, for all $w \in W$ then the proposed N–Q network efficiency measure (9) and the L–M measure (8) are one and the same.*

Proof Let n be the number of nodes in G. Hence, the total number of O/D pairs, n_W, is equal to $n(n-1)$ given the assumption that there exist positive demands for all pairs of nodes in G. Furthermore, by assumption, we have that $d_w = 1, \ \forall w \in W, w = (i, j)$, and $d_{ij} = \lambda_w$, where $i \neq j, \ \forall i, j \in G$. Then the L–M measure (8) becomes as follows:

$$E(G) = \frac{1}{n(n-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}} = \frac{\sum_{i \neq j \in G} \frac{1}{d_{ij}}}{n_W} = \frac{\sum_{w \in W} \frac{d_w}{\lambda_w}}{n_W} = \mathcal{E}(G, d), \tag{10}$$

and the conclusion follows. □

Note that, according to (6), λ_w is the value of the cost of the minimum or "shortest" used paths for O/D pair w and d_{ij}, according to [8], is the shortest path length (the geodesic distance) between nodes i and j. Therefore, the assumption that $d_{ij} = \lambda_w$ is not unreasonable given that $w = (i, j)$ and that there is assumed to be a positive demand between i and j. Our measure, however, is a more general measure since it also captures flows and behavior on the network, according to Definition 3.

Furthermore, we note that in the L–M measure, there is no information regarding the demand for each O/D pair. Therefore, $n(n-1)$ can be interpreted as the total possible number of O/D pairs regardless of whether there exists a demand for a pair of nodes or not. However, because the N–Q measure is an average network efficiency measure, it does not make sense to count a pair of nodes which has no associated demand in the computation of the network efficiency. Therefore, the number of O/D pairs, n_W, is more appropriate as a divisor in the N–Q measure than $n(n-1)$. Of course, if there is a positive associated demand between all pairs of nodes in the network then $n_W = n(n-1)$.

In this paper, when comparing the results obtained via the N–Q measure to those obtained by the L–M measure, we will always assume that $d_{ij} = \lambda_w$, where $w = (i, j)$, which makes sense also in practice, given equilibrium conditions (6); in other words, d_{ij} is the value of the shortest path connecting nodes i and j.

In addition, we recall (see also [10]) that the importance of network components according to the L–M measure is defined as follows:

Definition 4 (*Importance of a network component according to the L–M Measure*) The importance of a network component $g \in G$, $\bar{I}(g)$, is measured by the network efficiency drop, determined by the L–M measure, after g is removed from the network:

$$\bar{I}(g) = \triangle E = E(G) - E(G-g), \tag{11}$$

where $G - g$ is the resulting network after component g is removed from network G.

We note, however, that for all the examples in the paper of [10], the authors reported the importance and the ranking of network components in terms of the relative drop in efficiency. Hence, in order to make the results obtained when using the L–M measure directly comparable to those obtained from the N–Q measure, in the subsequent sections, we adopt the following formula when presenting the importance and the ranking of network components according to the L–M measure:

$$\bar{I}(g) = \frac{\triangle E}{E(G)} = \frac{E(G) - E(G-g)}{E(G)}. \tag{12}$$

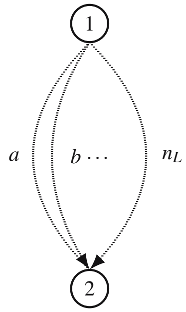

Fig. 1 Special network structure for networks associated with Theorem 3

Similarly, the importance of network components from the N–Q measure is defined as follows ([15]):

Definition 5 (*Importance of a network component according to the N–Q measure*) The importance of a network component $g \in G$, $I(g)$, is measured by the relative network efficiency drop, determined by the N–Q measure, after g is removed from the network:

$$I(g) = \frac{\triangle \mathcal{E}}{\mathcal{E}} = \frac{\mathcal{E}(G, d) - \mathcal{E}(G - g, d)}{\mathcal{E}(G, d)}, \tag{13}$$

where $G - g$ is the resulting network after component g is removed from network G.

The elimination of a link is treated in the N–Q measure by removing that link while the removal of a node is managed by removing the links entering and exiting that node. In the case that the removal results in no path connecting an O/D pair, we simply assign the demand for that O/D pair to an abstract path with a cost of infinity. Hence, our measure is well-defined even in the case of disconnected networks. Notably, [8] and [10] also mentioned this important characteristic which gives the L–M measure an attractive property over the measure used for the small-world network model (cf. [24]).

In the following theorem, we establish a special relationship between the importance of a network component determined by the L–M measure and that determined by the N–Q measure for networks with a single origin/destination (O/D) pair, connected by paths consisting of single links as in Fig. 1.

Theorem 3 *For a network G with a single O/D pair of nodes $w = (1, 2)$, and connected by n_L paths consisting of single links (cf. Fig. 1) with a given demand d_w for the O/D pair, the importance of a network component according to the L–M measure (defined in (12)) is equal to that obtained via the N–Q measure (defined in (13)).*

Proof Let d_w and λ_w be, respectively, the demand and the equilibrium disutility for $w = (1, 2)$. According to the definition of the efficiency measures, $E(G)$ and $\mathcal{E}(G, d)$, we know that for this special network that $n = 2$ and $n_W = 1$. □

Consider the importance of a network component, say, component g, in the network G. The same result can be obtained for any network component. Let $\hat{\lambda}_w$ be the equilibrium disutility after the removal of component g. By referring to (12), we have:

$$\bar{I}(\text{component } g) = \frac{\triangle E}{E} = \frac{\frac{\frac{1}{\lambda_w}}{2} - \frac{\frac{1}{\hat{\lambda}_w}}{2}}{\frac{\frac{1}{\lambda_w}}{2}} = \frac{\frac{1}{\lambda_w} - \frac{1}{\hat{\lambda}_w}}{\frac{1}{\lambda_w}}. \tag{14}$$

By referring to (13), we have:

$$I(\text{component } g) = \frac{\triangle\mathcal{E}}{\mathcal{E}} = \frac{\frac{\frac{d_w}{\lambda_w}}{1} - \frac{\frac{d_w}{\hat{\lambda}_w}}{1}}{\frac{\frac{d_w}{\lambda_w}}{1}} = \frac{\frac{1}{\lambda_w} - \frac{1}{\hat{\lambda}_w}}{\frac{1}{\lambda_w}}. \tag{15}$$

A comparison of (14) and (15) yields the equivalence.

Hence, in the case of the special network depicted in Fig. 1, although the N–Q and the L–M measures do not yield identical measures of efficiency $\mathcal{E}$ and E, respectively, the importance value obtained by either of these measures (and, hence, also the importance rankings) are identical.

For illustration purposes, we now provide a simple example.

A simple network example

Consider a network with special structure as depicted in Fig. 1 where $n_L = 2$. The network consists of two links, denoted, for simplicity, by a and b, respectively. The user cost link functions are assumed to be given by:

$$c_a(f_a) = f_a + 5, \quad c_b(f_b) = 2f_b + 16.$$

Due to the special network structure, we let $p_1 = a$ and $p_2 = b$, so that $x_{p_1} = f_a$ and $x_{p_2} = f_b$. Assume now that $w = (1, 2)$ and that the demand $d_w = 10$.

It is straightforward to determine (cf. (6)) that the equilibrium path flow pattern is $x^*_{p_1} = 10$ and $x^*_{p_2} = 0$, so that $C_{p_1}(x^*_{p_1}) = 15$ and $C_{p_2}(x^*_{p_2}) = 16$ (and, hence, only the minimal cost path, which is path p_1, is used, that is, has positive flow. Thus, $\lambda_w = 15$.

Let us compute the two network efficiency measures. The N–Q measure yields for this network: $\mathcal{E} = \frac{1}{n_W}\frac{d_w}{\lambda_w} = \frac{10}{15} = \frac{2}{3}$, whereas the L–M measure obtains the value $E = \frac{1}{n(n-1)}\frac{1}{d_{12}} = \frac{1}{2}\frac{1}{\lambda_w} = \frac{1}{30}$. Recall that in the L–M measure, if there is no path connecting a pair of nodes (i, j) then $d_{ij} \equiv \infty$ and, hence, $\frac{1}{d_{ij}} = 0$ in that case.

Let us investigate now the importance of the network component link b using both measures. According to the N–Q measure (see (15)) we have that:

$$I(b) = \frac{\frac{2}{3} - \frac{2}{3}}{\frac{2}{3}} = 0.$$

According to the L–M measure, in turn, we have that (cf. (14)):

$$\bar{I}(b) = \frac{\frac{1}{15} - \frac{1}{15}}{\frac{1}{15}} = 0.$$

Hence, as established in Theorem 3, both measures yield the identical importance value for the network component under consideration, link b. The importance value of link a, in turn, is equal to $\frac{7}{12}$ using either the N–Q or the L–M measure, as is also predicted by Theorem 3. The importance of nodes 1 and 2 yield a value of 1 using either measure.

In the subsequent sections, we apply the above measures and importance definitions to several critical infrastructure networks.

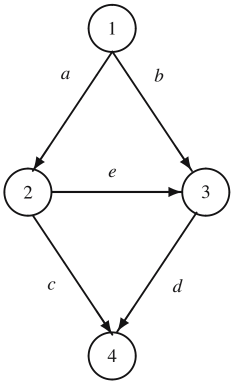

Fig. 2 The Braess network

4 An application of the N–Q measure and the L–M Measure to the Braess network

Consider the Braess Paradox example after the addition of a new link e and as depicted in Fig. 2 (see also [3] and [4]). There are four nodes: $1, 2, 3, 4$; five links: a, b, c, d, e; and a single O/D pair $w = (1, 4)$. There are, hence, three paths connecting the single O/D pair, which are denoted, respectively, by: $p_1 = (a, c)$, $p_2 = (b, d)$ and $p_3 = (a, e, d)$. The link cost functions are:

$$c_a(f_a) = 10f_a, \quad c_b(f_b) = f_b + 50,$$

$$c_c(f_c) = f_c + 50, \quad c_d(f_d) = 10f_d, \quad c_e(f_e) = f_e + 10.$$

We can also write down the path cost functions (cf. (5)) as follows:

$$C_{p_1}(x) = 11x_{p_1} + 10x_{p_3} + 50, \quad C_{p_2}(x) = 11x_{p_2} + 10x_{p_3} + 50,$$

$$C_{p_3}(x) = 10x_{p_1} + 21x_{p_3} + 10x_{p_2} + 10.$$

Recall that the Braess Paradox [3] demonstrated that for a fixed demand of $d_w = 6$ the addition of link e, which provides the users with the new path p_3, as in the network in Fig. 1, actually makes all users worse off since without the link e, the travel disutility and path costs are 83, whereas with the new link/path, the travel disutility and path costs go up for all users to 92!

Assume that the demand $d_w = 6$. By referring to [3] and [13], we know that the equilibrium path flow pattern is: $x^*_{p_1} = x^*_{p_2} = x^*_{p_3} = 2$. The equilibrium disutility is $\lambda_w = 92$. The N–Q efficiency for this range of demand is $\mathcal{E} = 0.0652$, whereas the L–M efficiency is $E = 0.0152$.

The importance and the rankings of the links and the nodes are given, respectively, in Tables 1 and 2.

4.1 Discussion

From the above link importance ranking results, we see that the links identified as the most important links according to the N–Q measure, that is, links a and d are ranked the least important according to the L–M measure. On the other hand, link e which is ranked

Table 1 Importance and ranking of the links in the Braess network

Link	Importance value from the N–Q measure	Importance ranking from the N–Q measure	Importance value from the L–M measure	Importance ranking from the L–M measure
a	0.2069	1	0.1056	3
b	0.1794	2	0.2153	2
c	0.1794	2	0.2153	2
d	0.2069	1	0.1056	3
e	−0.1084	3	0.3616	1

Table 2 Importance and ranking of the nodes in the Braess network

Node	Importance value from the N–Q measure	Importance ranking from the N–Q measure	Importance value from the L–M measure	Importance ranking from the L–M measure
1	1.0000	1	N/A	N/A
2	0.2069	2	0.7635	1
3	0.2069	2	0.7635	1
4	1.0000	1	N/A	N/A

least important according to the N–Q measure, is ranked most important according to the L–M measure. Since the addition of link e causes the Braess Paradox when demand is equal to 6 [3], it will, obviously, be detrimental to the network performance, which is clearly shown by the negative importance value of link e obtained via the N–Q measure. The fact that link e is ranked as the most important link in the L–M measure is unreasonable.

Moreover, we note that, in the above node ranking results, the importance values (and, hence, their rankings) of nodes 1 and 4 are not defined for the L–M measure. This is due to the fact that the cost functions of the links a and d are solely dependent on the flow on the respective link (and do not have any fixed cost terms). Take node 1 for example: the removal of node 1 is treated by removing links a and b as discussed in Sect. 3. But the cost on link d will be zero because of the cost structure on the link, which makes the L–M measure not defined. However, the N–Q measure is still well-defined with the removal of nodes 1 and 4.

5 A "coupled" Braess network and the importance identification of its nodes and links

Consider now the network depicted in Fig. 3 in which there are seven nodes: 1, 2, 3, 4, 5, 6, and 7; 10 links: $a, b, c, d, e, g, h, l, m, k$; and two O/D pairs $w_1 = (1, 4)$ and $w = (1, 7)$. There are, hence, three paths available for each O/D pair; for O/D pair $w_1 : p_1 = (a, c)$, $p_2 = (b, d)$, and $p_3 = (a, e, d)$ and for O/D pair $w_2 : p_4 = (g, k)$, $p_5 = (h, l)$, and $p_6 = (g, m, l)$.

The link cost functions are given by:

$$c_a(f_a) = 10 f_a, \quad c_b(f_b) = f_b + 50, \quad c_c(f_c) = f_c + 50, \quad c_d(f_d) = 10 f_d, \quad c_e(f_e) = f_e + 10,$$

$$c_g(f_g) = 10 f_g, \quad c_h(f_h) = f_h + 50, \quad c_k(f_k) = f_k + 50, \quad c_l(f_l) = 10 f_l, \quad c_m(f_m) = f_m + 10.$$

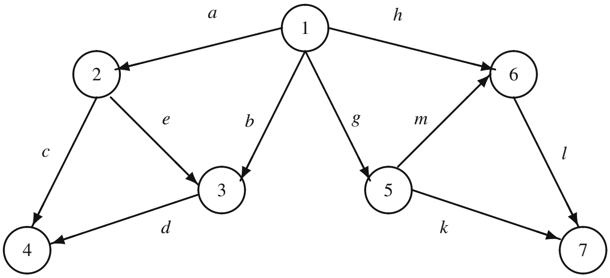

Fig. 3 A coupled Braess network

Table 3 Importance and the ranking of the links in the coupled Braess example

Link	Importance value from the N–Q measure	Importance ranking from the N–Q measure	Importance value from the L–M measure	Importance ranking from the L–M measure
a	0.1030	3	0.0928	2
b	0.0000	5	0.0470	4
c	0.0000	5	0.0470	4
d	0.1030	3	0.0928	2
e	0.0547	4	−0.0307	6
g	0.1301	1	0.0459	5
h	0.1131	2	0.0929	3
k	0.1131	2	0.0929	3
l	0.1301	1	0.0459	5
m	−0.0682	6	0.1557	1

The demands for the O/D pairs are: $d_{w_1} = 2$ and $d_{w_2} = 6$. The equilibrium path flow solution is:

$$x^*_{p_1} = x^*_{p_2} = 0, \quad x^*_{p_3} = 2,$$

$$x^*_{p_4} = x^*_{p_5} = x^*_{p_6} = 2,$$

with the travel disutilities, which correspond to the minimal cost used paths for each O/D pair, given by:

$$\lambda_{w_1} = 52, \quad \lambda_{w_2} = 92.$$

The N–Q efficiency is $\mathcal{E} = 0.0518$, whereas the L–M efficiency is $E = 0.0101$.

The importance and the rankings of the links and the nodes are given, respectively, in Tables 3 and 4.

5.1 Discussion

The N–Q measure and the L–M measure show quite different results regarding the importance rankings of the links and the nodes in the case of the coupled Braess network. Strikingly,

Table 4 Importance and ranking of the nodes in the coupled Braess example

Node	Importance value from the N–Q measure	Importance ranking from the N–Q measure	Importance value from the L–M measure	Importance ranking from the L–M measure
1	1.0000	1	N/A	N/A
2	0.1030	5	0.3750	1
3	0.1030	5	0.3750	1
4	0.3710	3	N/A	N/A
5	0.1301	4	0.3281	2
6	0.1301	4	0.3281	2
7	0.6290	2	N/A	N/A

link m is ranked as the most important link by the L–M measure while it is the least important according to the N–Q measure. By noting that link m causes the Braess Paradox in the right-hand-side network in Fig. 3, the result obtained for the N–Q measure is, clearly, more reasonable. Furthermore, by taking demands and flows into consideration, the N–Q measure ranks links g and l the most important while the L–M measure only places these links in fifth place in the importance ranking. On the other hand, links a and d are the second most important links according to the L–M measure while they are in third place according to the N–Q measure. This ranking difference is due to the fact that links g and l carry much higher flows as compared to links a and d. Therefore, it is reasonable to evaluate them as more important links and this is what the N–Q measure does. In addition, for the same reason as in the Braess network example, the L–M measure for the network without nodes 1, 4, and 7 is not defined. Hence, the importance value of these nodes and their ranking cannot be determined. However, the N–Q measure for the network without these nodes is well-defined.

6 An electric power supply chain example

In this section, we consider the application of the N–Q network performance/efficiency measure to another example of "critical infrastructure," that of electric power generation and distribution networks, in the form of an electric power supply chain problem. The example is adapted from Example 1 in [17], where the notation and model are fully described. In the example, the electric power supply chain network consists of one power generator, three power suppliers, one transmission provider, and one demand market as depicted in Fig. 4. The links in the first network in Fig. 4 are labeled as a, b, etc., for subsequent reference. The supernetwork representation which allows the transformation of the electric power supply chain network (see [25] and [17]) into a transportation network equilibrium problem is also depicted in Fig. 4.

The power generating cost function for the power generator is given by:

$$f_1(q_1) = 2.5q_1^2 + 2q_1.$$

The transaction cost functions faced by the power generator and associated with transacting with the power suppliers: 1, 2, 3 are given, respectively, by:

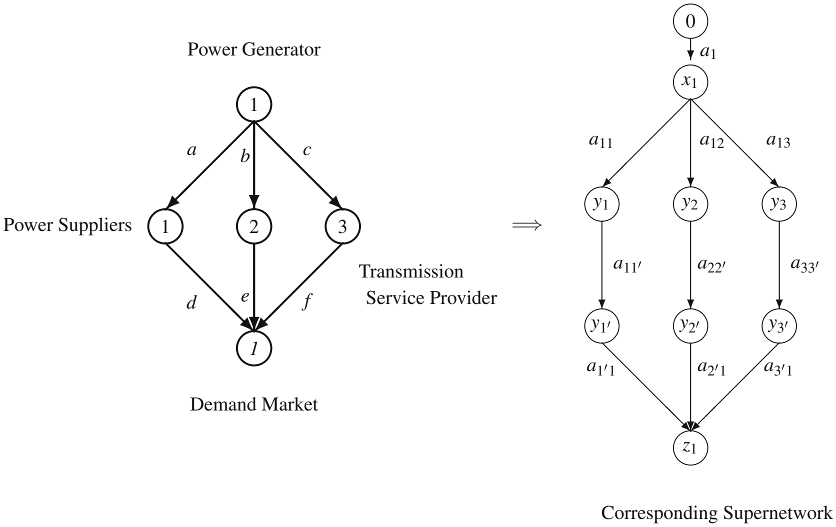

Fig. 4 Electric power supply chain network and the corresponding supernetwork

$$c_{11}(q_{11}) = .5q_{11}^2 + 3.5q_{11}, \quad c_{12}(q_{12}) = .5q_{12}^2 + 2.5q_{12},$$

$$c_{13}(q_{13}) = .5q_{13}^2 + 3.5q_{13}.$$

The operating costs of the three power suppliers, in turn, are given, respectively, by:

$$c_1(q_{11}) = .5q_{11}^2, \quad c_2(q_{12}) = .5q_{12}^2, \quad c_3(q_{13}) = .5q_{13}^2.$$

The unit transaction costs associated with transacting between the power suppliers and the demand market are:

$$\hat{c}_{11}(q_{11}) = q_{11} + 1, \quad \hat{c}_{21}(q_{21}) = q_{21} + 5, \quad \hat{c}_{31}(q_{31}) = q_{31} + 10.$$

In the corresponding supernetwork representation, there are 9 nodes and 10 links with 1 O/D pair denoted by $w_1 = (0, z_1)$. There are three paths connecting w_1:

$$p_1 = (a_1, a_{11}, a_{11'}, a_{1'1}), \quad p_2 = (a_1, a_{12}, a_{22'}, a_{2'1}), \quad p_3 = (a_1, a_{13}, a_{33'}, a_{3'1}).$$

The demand at the demand market is denoted by d_{w_1}.

We will present the importance and the ranking of the individual links and nodes for this example using the N–Q measure. We conduct the analysis in the context of the supernetwork (which represents the transportation network equilibrium transformation; see [17]. We then translate the results back to the first network.

Recall that (cf. Fig. 4), as discussed in [17], $c_{a_1} = \frac{\partial f_1}{\partial q_1}$, $c_{a_{gs}} = \frac{\partial c_{gs}}{\partial q_{gs}}$; for $gs = 11, 12, 13$; $c_{a_{ss'}} = \frac{\partial c_s}{\partial q_{1s}}$, for $s = 1, 2, 3$, and $c_{a_{s'1}} = \hat{c}_{s1}$, for $s = 1, 2, 3$. The cost on a path $p_s \equiv p_{1ss'1}$, for $s = 1, 2, 3$ joining node 0 to node z_1 in the above supernetwork representation is:

$$C_{p_s} \equiv C_{p_{1ss'1}} = c_{a_1} + c_{a_{1s}} + c_{a_{ss'}} + c_{a_{s'1}}, \quad s = 1, 2, 3.$$

Assume now that the demand $d_{w_1} = 4$. By a simple calculation, we know that $x_{p_1}^* = 2.4444$, $x_{p_2}^* = 1.4444$, and $x_{p_3}^* = 0.1112$. The equilibrium disutility is then $\lambda_{w_1} = 33.8332$. The N–Q efficiency is $\mathcal{E} = 0.1182$. The importance and the rankings of the links and the nodes are given, respectively, in Tables 5 and 6.

Table 5 Importance and ranking of the links for the electric power supply chain network

Link	Importance value from the N–Q measure	Importance ranking from the N–Q measure
a	0.1212	1
b	0.0856	2
c	0.0049	3
d	0.1212	1
e	0.0856	2
f	0.0049	3

Table 6 Importance and ranking of the nodes of the electric power supply chain network

Node	Importance value from the N–Q measure	Importance ranking from the N–Q measure
Power Generator 1	1.0000	1
Power Supplier 1	0.1212	2
Power Supplier 2	0.0856	3
Power Supplier 3	0.0049	4
Demand Market 1	1.0000	1

6.1 Discussion

In this example, we note that links a and d are the most important links. The power generator and the demand market are, in turn, the most important nodes, which is reasonable. Power supplier 1 is ranked as the second most important node. By observing the equilibrium path flows, we can see that path p_1, which includes links a and d and power supplier 1, carries the largest amount of flow. Therefore, given the fact that three paths have the same equilibrium disutility, the removal of links a or d will have the most severe impact on the electric power supply chain network efficiency. The removal of power supplier 1 will have the largest impact on the network efficiency only second to that of the removal of power generator 1 or the demand market. Furthermore, at the imposed demand, link b is more important than link c and link e is more important (that is, is ranked higher) than link f.

7 Summary and conclusions

In this paper, we have demonstrated how a network performance/efficiency measure, referred to as the N–Q measure, that captures demands, flows, as well as the behavior of users of the network, and incurred costs, can be applied to assess the efficiency of critical infrastructure networks as well as the importance and ranking of network components, that is, the nodes and links. We obtained new theoretical results and also demonstrated that the new measure provides more realistic assessments of the performance of critical infrastructure networks as well as network component importance identification as compared to an existing measure, the L–M measure. Applications to transportation networks, including the Braess network and the coupled Braess network, with relevance to the Internet were also given. Finally, an application of the N–Q measure to an electric power supply chain network example was included.

Future research will apply the N–Q measure to other critical infrastructure networks, including financial networks (cf. Liu and Nagurney [11]), with a specific focus on large-scale critical infrastructure networks.

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